O. N. R. RESEARCH MEMORANDUM NO. 10

COMPUTATIONAL THEORY OF LINEAR PROGRAMMING

I: THE "BOUNDED VARIABLES" PROBLEM

by

A. Charnes and C. E. Lemke

Graduate School of Industrial Administration

Carnegie Institute of Technology

O. N. R. RESEARCH MEMORANDUM No. 10

COMPUTATIONAL THEORY OF LINEAR PROGRAMMING

I: THE "BOUNDED VARIABLES" PROBLEM

by

A. Charnes and C. E. Lemke

January 7, 1954

Graduate School of Industrial Administration

Carnegie Institute of Technology

This paper was written as part of the project, "The Planning and Control of Industrial Operations" under grant from the Office of Naval Research. The paper is released for limited circulation in order to facilitate discussion, and invite criticism. It is wholly tentative in character and should not be referred to in publication without the permission of the authors.

COMPUTATIONAL THEORY OF LINEAR PROGRAMMING I: THE "BOUNDED VARIABLES" PROBLEM

Introduction

This paper is the first of a series1/ devoted to extension and sharpening of the present methods of linear programming so as to make practicable the calculation of programs involving many more restrictions than can now be conveniently handled. These extensions are developments on the technique first employed by Charnes (ref. [1]) in reducing every linear programming problem to one with a bounded set of solutions.

We here develop an "extended" simplex method for the "bounded variables" problem, e. g., any linear programming problem in which every variable entering is constrained to lie between an upper and a lower bound. This includes as a special case those problems in which not all are so constrained since we can prescribe arbitrarily large (or small) upper (or lower) bounds for those unconstrained. The advantage achieved with the extended method is reduction in size of the computational tableau by suppression of the inequalities expressing the bounds.2/

Examples of such constraints occur repeatedly in policy limitations on items of inventory, in market limitations of saleable amounts of products, in production and delivery requirements (cf. ref. [2]), etc. Rather than expound on such examples here (which require individual development for adequate coverage) we shall attempt to present, in each paper, an example seemingly radically divorced from production or industrial connotations in order to gain one of the most important by-products of mathematical formalism, namely,

---

1/ We wish to thank W. W. Cooper for suggestions, stimulation and encouragement of this series.

2/ We understand that G. B. Dantzig and associates at Rand Corporation had developed a similar method (suggested by Charnes, Ref. [1]) in connection with unspecified, security-restricted problems. Their results are as yet unavailable.

the recognition of an analogous problem in a completely different field from which new insights may be gained into the original problem. The example presented in this paper is that of plastic collapse of structures (frames).

We first review the simplex method in terms suitable for either the "normal" procedure or the "modified" procedure or the "dual" method (ref. [3], [4] and [5]).

The Simplex Method.

The general linear programming problem may be put into the following form, which we shall refer to throughout as the "simplex" problem:

$$\text{Maximize:} \quad Z_o = \sum_{j=1}^{n} c_j \rho_j$$

$$\text{where:} \quad b_i = \sum_{j=1}^{n} a_{ij} \rho_j \; ; \qquad i = 1, \ldots, m$$

$$\text{and:} \quad \rho_j \geq 0 \; ; \; j = 1, \ldots, n.$$

It will be convenient, both for the review of the simplex method and for some simple proofs, to restate the problem in vector notation as:

$$\text{Maximize:} \quad Z_o = \sum_{j=1}^{n} \rho_j c_j \, ,$$

[1] $$\text{where:} \quad P_o = \sum_{j=1}^{n} \rho_j P_j \, , \quad \text{and } \rho_j \geq 0 \; ; \; j = 1, \ldots, n.$$

Thus, P_o, P_1, ..., P_n are vectors in the Euclidean vector space V_m of mx1 matrices. Throughout we shall refer to c_j as "the scalar corresponding to the vector P_j".

Suppose that a set B_m of linearly independent vectors, call them a_1, a_2, ..., a_m are selected from the vectors P_1, ..., P_n. Thus, B_m is a basis of V_m and we may write uniquely:

[2] $$P_o = \sum_{i=1}^{m} \rho_i a_i .$$

If the m values ρ_i satisfy $\rho_i \geq 0$; $i = 1, \ldots, m$ they then form a solution to the system [1]. When the ρ_i also satisfy $\rho_i > 0$; $i = 1, \ldots, m$, the solution is referred to as a basic solution. It has been shown (ref. [1]) that the problem can be so modified that when one has a solution to [1] of the form [2] the solution is always a basic solution. Hence we may make the initial assumption that the original set of vectors $P_o, P_1, \ldots, P_n$ has this property, which may be characterized by the statement that the vector P_o is linearly independent of any m-1 vectors chosen from among $P_1, \ldots, P_n$.

With this assumption, the simplex method may be described as one which proceeds from basic solution to basic solution until a maximal basic solution is found, i. e., one which yields the maximum value for Z_o. We shall review the procedure for going from one basic solution to another by the simplex method. Thus, suppose one has located a basis B_m such that, expressing P_o as in [2] one has $\rho_i > 0$; $i = 1, \ldots, m$. Let c_{r_i} be the scalar corresponding to the vector a_i.

We introduce the unique vectors a^i; $i = 1, \ldots, m$ satisfying:

[3] $$a_i{'}a^j = \delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} ; \quad \text{for } i, j = 1, \ldots, m.$$

We shall call the vectors a^i the "dual vectors" for the set a_i. If A is the matrix whose ith column is a_i then the jth row of A^{-1} is a^j. Having the dual vectors and denoting the inner product of vectors x and y by x'y any vector x in V_m can be expressed by:

[4] $$x = \sum_{i=1}^{m} (x{'}a^i)a_i.$$

In particular, we have:

[5] $$P_o = \sum_{i=1}^{m} (P_o{'}a^i)a_i,$$

[6] $$P_j = \sum_{i=1}^{m} (P_j{'}a^i)a_i \ ; \quad j = 1, \ldots, n.$$

Having the computed data $P_o{}'a^i$ and $P_j{}'a^i$, the first step in the procedure is to compute the quantities:

$$[7] \qquad Z(P_j) = \sum_{i=1}^{m} (P_j{}'a^i)c_{r_i} - c_j \quad ; \quad j = 1, \ldots, n.$$

$Z(P_j)$ is formed by replacing each a_i by its corresponding scalar c_{r_i} in the expression [6] for P_j and then subtracting the scalar corresponding to P_j.

When the n quantities [7] are all non-negative, one may demonstrate that the problem is finished: the basic solution is a maximal one. When at least one of the quantities [7] is negative, a new basic solution may be secured which increases the functional Z_o. This is accomplished by forming a new basis $B_m{}'$ from B_m by replacing one of the a_i by another vector. The replacing vector may be any P_k for which $Z(P_k) < 0$. To decide upon the vector to be replaced one finds the value of i, say r, such that:

$$[8] \qquad \theta = (P_o{}'a^r)/(P_k{}'a^r) = \min_i.(P_o{}'a^i)/(P_k{}'a^i) \quad \text{for } P_k{}'a^i > 0.$$

The following facts may then be demonstrated (ref. [6]):

i) The value r which yields θ is unique.

ii) The set $B_m{}'$ consisting of $a_1, \ldots, a_{r-1}, P_k, a_{r+1}, \ldots, a_m$ is a basis and again yields a basic solution.

iii) The new basic solution yields a larger value of the functional Z_o.

Having decided upon the values of k and r, when the vectors $P_o, P_1, \ldots, P_n$ are expressed in terms of the new basis $B_m{}'$, the transition to the new basic solution may be considered complete. (There is a simple algorithm for this transition which will be presented later).

One may then show that in a finite number of steps one will obtain a maximal basic solution, namely one for which all of the quantities $Z(P_j)$ are non-negative.

It is usual, both for computational and explanatory purposes, to construct a computational tableau which exhibits the computations pertaining to a given stage and at the same time clarifies, for the computer, the mechanics of proceeding to the next stage.

Computational Tableau

Corresponding scalars →			c_1	...	c_k	...	c_n
↓	B_m	P_o	P_1	...	P_k	...	P_n
c_{r_1}	a_1	$P_o{}'a^1$	$P_1{}'a^1$	...	$P_k{}'a^1$	...	$P_n{}'a^1$
⋮	⋮	⋮	⋮		⋮		⋮
c_{r_s}	a_s	$P_o{}'a^s$	$P_1{}'a^s$	...	$P_k{}'a^s$	...	$P_n{}'a^s$
⋮	⋮	⋮	⋮		⋮		⋮
c_{r_m}	a_m	$P_o{}'a^m$	$P_1{}'a^m$	...	$P_k{}'a^m$	...	$P_n{}'a^m$
$Z(P_j)$ →		z_o	$Z(P_1)$	...	$Z(P_k)$	...	$Z(P_n)$

We shall refer to such a tableau as an "mxn tableau" although there are a few additional rows and columns, and shall likewise refer to the problem as an "mxn problem".

The Bounded Variables Problem

First an illustration. The following problem arises in the theory of plastic collapse for structures (refs. [7], [8]).

Consider a structure composed of elastic, perfectly plastic beams joined together rigidly. We are interested in plastic collapse of this structure under the action of concentrated applied forces. We assume that each beam is homogeneous so that the same yield conditions apply to any cross-section of the same beam.

Consider a set of numbers $p_1, \ldots, p_m$ which represent the magnitudes of external forces which are to be applied to the structure. The structure

will be in equilibrium when these forces are balanced by the reactions at the supports. These reactions, together with the applied forces, give rise to a system of internal forces which determine the bending moment at every point of the system (the effects of shear and axial forces can be neglected here).

For a given cross-section there will be a maximum and a minimum bending moment which the cross-section can carry due to the assumption of perfect plasticity. Thus, at each cross-section the bending moment M will be subject to an inequality $-M^o \leq M \leq M^o$. M^o is the "fully plastic moment" or the moment for which plastic flow would first begin. Now, the bending moment distribution will be linear along each member. Hence, we need only be concerned with those points of the structure where plastic yielding will first occur. These points are finite in number and depend only on the structure and on the points of application of the forces. They are points including those for which the bending moment has a turning point, as well as the ends of the beams, or points where a support is present, or points at which one of the forces is applied. These points are called "test stations". Thus, at each test station the bending moment M_j is subject to the restrictions:

[9] $$-M_j^o \leq M_j \leq M_j^o \quad ; \quad j = 1, \ldots, n.$$

Now, considering the forces $p_1, \ldots, p_m$ as a <u>base</u> load, we apply instead the loads $\rho p_1, \ldots, \rho p_m$. Then, besides the restrictions [9], the bending moments M_j will be subject to the equilibrium conditions:

[10] $$\sum_{j=1}^{n} b_{ij} M_j = \rho p_i \quad ; \quad i = 1, \ldots, m,$$

where the b_{ij} depend on the structure and on the points of application of the forces.

We seek the largest value $\bar{\rho}$ of ρ permissible in order that the structure remain in equilibrium under the action of the loads ρp_i.

We make the following transformations and notational changes: Let $x_j = (M_j/M_j^o) + 1$; $x_{n+1} = \rho$; $a_{ij} = b_{ij}M_j^o$; and $b_i = \sum_{j=1}^{n} a_{ij}$. The problem then becomes:

$$\text{Maximize:} \quad Z_o = x_{n+1}$$

$$\text{where:} \quad b_i = \sum_{j=1}^{n} a_{ij}x_j + (-p_i)x_{n+1} \quad ; \quad i = 1, \ldots, m,$$

$$\text{and:} \quad 0 \leq x_j \leq 2 \quad ; \quad j = 1, \ldots, n.$$

This is an example of the following problem which we shall state in vector form and which we shall call the "bounded variables" problem:

$$\text{Maximize:} \quad Z_o = \sum_{j=1}^{n} \rho_j c_j \; ,$$

$$\text{where:} \quad P_o = \sum_{j=1}^{n} \rho_j P_j \; , \quad \text{and} \quad 0 \leq \rho_j \leq b_j \; ; \; j = 1, \ldots, n.$$

We shall set up and treat this problem as a simplex problem. In this manner we shall have an $(m+n)x(2n)$ problem. Such a problem would ordinarily employ $(m+n)x(2n)$ tableaux. We shall show, however, that it will be sufficient to employ mxm tableaux; that is, that the problem can be handled in the space V_m of the vectors P_j.

If we introduce the new variables x_j defined by:

[11] $$\rho_j + x_j = b_j \quad ; \quad j = 1, \ldots, n,$$

we may phrase the problem as follows in simplex form:

$$\text{Maximize:} \quad Z_o = \sum_{j=1}^{n} \rho_j c_j \; ,$$

$$\text{where:} \quad P_o = \sum_{j=1}^{n} \rho_j P_j \; ,$$

$$b_j = \rho_j + x_j \quad \text{and} \quad \rho_j, x_j \geq 0 \; ; \quad j = 1, \ldots, n.$$

We wish to define vectors in V_{m+n}. These will be designated by putting bars above the letters, as $\bar{z}$. At times we shall use the notation $\bar{z} = \begin{bmatrix} x \\ y \end{bmatrix}$, where x is a vector in V_m and y is a vector in V_n.

We introduce the following vectors in V_n:

$$b = \begin{pmatrix} b_1 \\ \vdots \\ \cdot \\ \cdot \\ b_n \end{pmatrix}, \qquad Q_j = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \text{-(jth place)} \quad ; \quad j = 1, \ldots, n.$$

And the following vectors in V_{m+n}:

$$\overline{P}_o = \begin{bmatrix} P_o \\ b \end{bmatrix}, \quad \overline{P}_j = \begin{bmatrix} P_j \\ Q_j \end{bmatrix}, \quad \overline{Q}_j = \begin{bmatrix} 0 \\ Q_j \end{bmatrix} \quad ; \quad j = 1, \ldots, n.$$

The $(m+n)x(2n)$ problem may then be phrased as:

[12]

$$\text{Maximize:} \quad Z_o = \sum_{j=1}^{n} \rho_j c_j ,$$

$$\text{where:} \quad \overline{P}_o = \sum_{j=1}^{n} \rho_j \overline{P}_j + \sum_{j=1}^{n} x_j \overline{Q}_j \quad ; \quad \rho_j, x_j \geq 0 \quad ; \quad j = 1, \ldots, n.$$

Suppose that a basic solution for this problem has been found. This implies that a basis B_{m+n} of V_{m+n} has been chosen from among the vectors $\overline{P}_1, \ldots, \overline{P}_n, \overline{Q}_1, \ldots, \overline{Q}_n$. Recall that the simplex method requires that each of these 2n vectors, together with $\overline{P}_o$, be expressed in terms of the basis B_{m+n}. We show that the coefficients involved can be obtained by merely expressing the vectors $P_1, \ldots, P_n$ in terms of a certain basis B_m of V_m.

To do this let us first note some properties of an arbitrary basis B_{m+n} chosen as above.

i) The number s of vectors $\overline{P}_j$ in B_{m+n} satisfies $m \leq s \leq n$.

ii) For each j, either $\overline{P}_j$ or $\overline{Q}_j$ or both are in B_{m+n}.

iii) By i and ii there are precisely m values of j such that both $\overline{P}_j$ and $\overline{Q}_j$ are in B_{m+n}. Furthermore, the set B_m of the m vectors P_j such that both $\overline{P}_j$ and $\overline{Q}_j$ are in B_{m+n} is a basis of V_m.

In particular, suppose that B_{m+n} yields a basic solution for the problem [12]. For notational ease we suppose that $\overline{P}_1, \ldots, \overline{P}_s$ are in B_{m+n} and

that $\overline{P}_i$ and $\overline{Q}_i$ are both in B_{m+n} for $i = 1, \ldots, m$. As a result we have that $\overline{Q}_{s+1}, \ldots, \overline{Q}_n$ complete the basis B_{m+n} and that $P_1, \ldots, P_m$ form a basis B_m of V_m.

Let $a^1, \ldots, a^m$ be the vectors dual to the basis B_m. Thus, we have:

[13] $$P_i{}'a^j = \delta_{ij} \quad ; \quad i, j = 1, \ldots, m.$$

Suppose that we have computed the quantities $P_o{}'a^i$ and $P_j{}'a^i$ satisfying:

[14] $$P_o = \sum_{i=1}^{m} (P_o{}'a^i)P_i$$

[15] $$P_j = \sum_{i=1}^{m} (P_j{}'a^i)P_i \quad ; \quad j = 1, \ldots, n.$$

By means of these quantities we may write down the expressions for $\overline{P}_o$, $\overline{P}_j$ for $s < j \leq n$ and $\overline{Q}_j$ for $m < j \leq s$ in terms of B_{m+n}: namely, for those vectors not already in the basis B_{m+n}. Thus, we have:

[15] $$\overline{P}_j = \overline{Q}_j + \begin{bmatrix} P_j \\ 0 \end{bmatrix} = \overline{Q}_j + \sum_{i=1}^{m} (P_j{}'a^i)[\overline{P}_i - \overline{Q}_i] \quad ; \quad s < j \leq n,$$

[16] $$\overline{Q}_j = \overline{P}_j - \begin{bmatrix} P_j \\ 0 \end{bmatrix} = \overline{P}_j - \sum_{i=1}^{m} (P_j{}'a^i)[\overline{P}_i - \overline{Q}_i] \quad ; \quad m < j \leq s,$$

[17] $$\overline{P}_o = \begin{bmatrix} P_o \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = \sum_{i=1}^{m} (P_o{}'a^i)[\overline{P}_i - \overline{Q}_i] + \sum_{j=1}^{n} b_j\overline{Q}_j =$$

$$= \sum_{i=1}^{m} (P_o{}'a^i)[\overline{P}_i - \overline{Q}_i] + \sum_{i=1}^{m} b_i\overline{Q}_i + \sum_{i=s+1}^{n} b_i\overline{Q}_i + \sum_{j+m+1}^{s} b_j[\overline{P}_j - \sum_{i=1}^{m} P_j{}'a^i(\overline{P}_i - \overline{Q}_i)] =$$

$$= \sum_{i=1}^{m} \phi_i\overline{P}_i + \sum_{i=1}^{m} (b_i - \phi_i)\overline{Q}_i + \sum_{j=m+1}^{s} b_j\overline{P}_j + \sum_{j=s+1}^{n} b_j\overline{Q}_j \quad ,$$

where we have set $\phi_i = P_o{}'a^i - \sum_{j=m+1}^{s} b_j(P_j{}'a^i)$ for $i = 1, \ldots, m$.

We recall now that the first step in applying the simplex method is to calculate the quantities $Z(\overline{P}_j)$ and $Z(\overline{Q}_j)$, obtained (see eqn. [7]) from the vector expressions [15] and [16] by replacing each of the basis vectors by its corresponding scalar.

Thus, in view of [15] and [16], these quantities become:

$$[18] \qquad Z(\overline{P}_j) = 0 + \sum_{i=1}^{m} (P_j{}'a^i)c_i - c_j = Z(P_j) \qquad ; \qquad s < j \leq n,$$

$$[19] \qquad Z(\overline{Q}_j) = c_j - \sum_{i=1}^{m} (P_j{}'a^i)c_i - 0 = -Z(P_j) \qquad ; \qquad m < j \leq s.$$

The values in [18] and [19] will determine whether or not the basic solution corresponding to the basis B_{m+n} is maximal. If it is not, one of these quantities will be negative and we may proceed to a new basic solution. In this case, having decided which vector will replace a vector in B_{m+n}, we need to find the vector which is to be replaced. For this we need the quantity θ (see eqn. [8]). The ratios competing for θ are found by means of equations [15], [16], and [17]. We need to distinguish two cases.

i) If $\overline{Q}_k$ for $m < k \leq s$ is to replace some vector in B_{m+n}:

$$\theta = \text{Min.}\left\{[\underset{i}{\text{Min.}}\ \phi_i/(-P_k{}'a^i) \text{ for } P_k{}'a^i<0],\ [\underset{i}{\text{Min.}}(b_i-\phi_i)/P_k{}'a^i \text{ for } P_k{}'a^i>0,\ b_k\right\}.$$

ii) If $\overline{P}_k$ for $s < k \leq n$ is to replace some vector in B_{m+n}:

$$\theta = \text{Min.}\left\{[\underset{i}{\text{Min.}}\phi_i/P_k{}'a^i \text{ for } P_k{}'a^i>0],\ [\underset{i}{\text{Min.}}(b_i-\phi_i)/(-P_k{}'a^i) \text{ for } P_k{}'a^i<0],\ b_k\right\}.$$

Having determined the vector to be replaced, one can complete the replacement and proceed to the new basic solution.

We have thus shown that essentially all of the computations in one stage of the (m+n)x(2n) problem may be handled by means of an mxn tableau. To complete the discussion of the bounded variables problem, we shall exhibit the computational tableau associated with one stage of the problem and list the rules of procedure to be followed in passing to the next stage.

Extended Tableau No. 1

b_j corresponding				b_1	..		b_{m+1}	..	b_s	b_{s+1}	..	b_n
scalars $\longrightarrow$				c_1	..	c_m	c_{m+1}	..	c_s	c_{s+1}	..	c_n
$\downarrow$	B_m	a P_o	b	P_1	..	P_m	P_{m+1}	..	P_s	P_{s+1}	..	P_n
c_1	P_1	ϕ_1	$b_1-\phi_1$	$P_1{}'a^1$	..	$P_m{}'a^1$	$P_{m+1}{}'a^1$	..	$P_s{}'a^1$	$P_{s+1}{}'a^1$	..	$P_n{}'a^1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
c_r	P_r	ϕ_r	$b_r-\phi_r$	$P_1{}'a^r$	..	$P_m{}'a^r$	$P_{m+1}{}'a^r$	..	$P_s{}'a^r$	$P_{s+1}{}'a^r$	..	$P_n{}'a^r$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
c_m	P_m	ϕ_m	$b_m-\phi_m$	$P_1{}'a^m$	..	$P_m{}'a^m$	$P_{m+1}{}'a^m$	..	$P_s{}'a^m$	$P_{s+1}{}'a^m$	..	$P_n{}'a^m$
$Z(P_j) \longrightarrow$			Z_o	$Z(P_1)$	..	$Z(P_m)$	$Z(P_{m+1})$	..	$Z(P_s)$	$Z(P_{s+1})$	..	$Z(P_n)$
$B_{m+n} \longrightarrow$				+, −	..	+, −	+	..	+	−	..	−

Explanations and Rules of Procedure:

1. The basis B_{m+n} is visualized in the tableau as follows:

   $\overline{P}_j$ is in B_{m+n} if the last entry in the column headed by P_j is a +.

   $\overline{Q}_j$ is in B_{m+n} if the last entry in the column headed by P_j is a −.

   Thus, both + and − at the bottom of the column headed by P_j indicates that both $\overline{P}_j$ and $\overline{Q}_j$ are in B_{m+n}.

2. For the procedure, one first checks the n values $Z(P_j)$ in the next to the last row. A maximum has been reached if both:

   $Z(P_j) \geq 0$ in all of those columns having + in the last entry;

   $Z(P_j) \leq 0$ in all of those columns having − in the last entry.

[This would mean that all of the quantities $Z(\overline{P}_j)$ and $Z(\overline{Q}_j)$ were non-negative.]

3. If some $Z(P_j)$ violates the above, we proceed to a new basic solution. Having decided upon the replacing vector, we need the value θ given in either i or ii on the preceding page. It is for this reason that the columns a and b under P_o and the first row were added to the tableau.

4. Having decided upon the replacing vector and the vector to be replaced we may pass on to the new basic solution and the new tableau. The following cases may arise:

i. The vector $\overline{Q}_k$ replaces the vector $\overline{P}_k$ in B_{m+n} for some k with $m < k \leq s$. This induces no change in the basis B_m. The only changes in the tableau in this case are the following:

a. The + under the column headed by P_k is changed to −.

b. The quantity $\phi_i' = \phi_i + b_k(P_k'a^i)$; $i = 1, \ldots, m$ replaces ϕ_i in the a and b columns under P_o. (see the expression for ϕ_i in eqn. 17).

c. The quantity $b_k Z(P_k)$ is added to Z_o. (Z_o, of course, need not be carried along. The maximum value of Z_o may be obtained at the end of the problem from eqn. 17.)

ii. The vector $\overline{P}_k$ replaces the vector $\overline{Q}_k$ in B_{m+n} for some k with $s < k \leq n$. This induces no change in B_m. The only changes in the tableau in this case are the following:

a. The − under the column headed by P_k is changed to +.

b. The quantity $\phi_i' = \phi_i - b_k(P_k'a^i)$; $i = 1, \ldots, m$ replaces ϕ_i in the a and b columns under P_o.

c. The quantity $b_k Z(P_k)$ is subtracted from Z_o.

iii. If either $\overline{P}_r$ or $\overline{Q}_r$ with $1 \leq r \leq m$ is replaced either by some $\overline{P}_k$ with $s < k \leq n$ or some $\overline{Q}_k$ with $m < k \leq s$ then P_k replaces P_r in the basis B_m to form a new basis B_m' and, as in the usual simplex procedure, the new tableau must be based upon the basis B_m'. The simplex algorithm for changing the tableau elements, which is included here for the sake of completeness, is as follows. The element $P_j'a^i$ is replaced by the element:

$$(P_j'a^i)' = P_j'a^i - (P_k'a^i/P_k'a^r)P_j'a^r \quad \text{for } j = 1, \ldots, n \text{ but } j \neq k,$$
$$\text{and for } i = 1, \ldots, m \text{ but } i \neq r,$$

$$(P_j'a^r)' = P_j'a^r/P_k'a^r \quad ; j = 1, \ldots, n,$$

$$(P_k'a^i)' = \begin{cases} 0 \text{ if } i \neq k \\ 1 \text{ if } i = k. \end{cases}$$

The n quantities $Z(P_j)$ are replaced by:

$$Z(P_j)' = Z(P_j) - [P_j'a^r/P_k'a^r]Z(P_k) \quad ; \quad j = 1, \ldots, n.$$

Under P_o the columns headed a and b are changed by replacing ϕ_i by:

$\phi_i' = \phi_i - \theta P_k' a^i$; for $i = 1, \ldots, m$, but $i \neq r$, and θ replaces ϕ_r in the rth place of the a column; $b_k - \theta$ replaces $b_r - \phi_r$ in the rth place of the b column.

Finally, P_k replaces P_r in the B_m column; c_k replaces c_r in the corresponding scalars column; and both + and - appear in the last entry of the P_k column, whereas the P_r column has a + if $\overline{Q}_r$ has been replaced in B_{m+n} or a - if $\overline{P}_r$ has been replaced.

This completes the discussion of the bounded variables problem.

As a final remark, it should be noted that when a change in the tableau (case iii above) is necessary, the modified simplex method (ref. [4]) may be employed, whereby only the vectors $(a^i)'$ dual to the new basis B_m' are computed from the old vectors a^i, and then the quantities $(P_j' a^i)'$ are obtained, <u>if needed</u>, by taking the scalar product of P_j and $(a^i)'$. The algorithm for obtaining the new dual vectors is:

$(a^r)' = (1/P_k' a^r) a^r$

$(a^i)' = a^i - (P_k' a^r)(a^r)'$ for $i = 1, \ldots, m$ but $i \neq r$.

References

[1] Charnes, A., "Optimality and Degeneracy in Linear Programming", Econometrica, Vol. 20, 2, April 1952, pp. 169.

[2] Charnes, A., Cooper, W. W., Mellon, B., "A Model for Programming and Sensitivity Analysis in an Integrated Oil Company", Econometrica, Forthcoming.

[3] Charnes, A., Cooper, W. W., Henderson, A., "Introduction to Linear Programming", John Wiley and Sons, 1953.

[4] Charnes, A., Lemke, C. E., "A Modified Simplex Method for Control of Round-off Error in Linear Programming", Proc. Assn. for Computing Machinery, 1952, p. 97.

[5] Lemke, C. E., "The Dual Method of Solving the Linear Programming Problem", Naval Research Logistics Quarterly, Jan. 1954.

[6] Dantzig, G. B., "Activity Analysis of Production and Allocation", Chapter XXI, Cowles Commission Monograph 13, New York, John Wiley and Sons, 1951.

References (contd.)

[7] Greenberg, H. J., Prager, W., "Limit Design of Beams and Frames", Proc. Amer. Soc. of Civil Engineers, Vol. 77, February, 1951.

[8] Charnes, A., Greenberg, H. J., "Plastic Collapse and Linear Programming", abstract presented at Summer Meeting of Amer. Math. Society, Sept. 1951.